MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2019

Calculator-free

Marking Key

Question 1 **(4 marks)**

Solution
$y = \int 2e^{3x} dx$ $y = \frac{2}{3}\int 3e^{3x} dx$ $y = \frac{2}{3}e^{3x} + c$ $\left(\frac{1}{3}, \frac{5e}{3}\right) \Rightarrow y = \frac{2}{3}e^{3x} + c$ $\frac{5e}{3} = \frac{2}{3}e^{1} + c$ $e = c$ $\therefore$ Equation of the curve that passes through the point $\left(\frac{1}{3}, \frac{5e}{3}\right)$ is $y = \frac{2}{3}e^{3x} + e$

Mathematical behaviours	Marks
• anti-differentiates using $\int f'(x)e^{f(x)}dx = e^{f(x)} + c$	1
• anti-differentiates using a factor of one-third	1
• substitutes the coordinates $\left(\frac{1}{3}, \frac{5e}{3}\right)$ into the anti-derivative function to determine the constant of integration correctly	1
• states the equation of the curve containing $\left(\frac{1}{3}, \frac{5e}{3}\right)$ correctly	1

Question 2(a) **(3 marks)**

Solution
Since $$\frac{d}{dx}(\cos x) = -\sin x$$ then with $$u = \cos x$$ we obtain $$\int_0^{\pi} \sin x \cos^6 x\, dx = -\int_1^{-1} u^6 \ du = \left[\frac{u^7}{7}\right]_{-1}^{1} = \frac{2}{7}.$$

Mathematical behaviours	Marks
• identifies the correct substitution	1
• integrates correctly	1
• evaluates the indefinite integral at the end points	1

Question 2(b) **(3 marks)**

Solution
Since $$\cos 2x = 2\cos^2 x - 1$$ it follows that $$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\cos x} = \frac{1}{2}\int_{\pi/3}^{\pi/2} \sec^2(x/2)\ dx = \left[\tan(x/2)\right]_{\pi/3}^{\pi/2} = \tan(\pi/4) - \tan(\pi/6) = 1 - \frac{1}{\sqrt{3}}.$$

Mathematical behaviours	Marks
• simplifies the integral to requiring the anti-derivative of $\sec^2(x/2)$	1
• integrates correctly	1
• evaluates the indefinite integral at the end points	1

Question 2(c) **(3 marks)**

Solution	
If we put $u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$ then we find that $\displaystyle\int_e^{e^2} \frac{dx}{x \ln x} = \int_1^2 \frac{du}{u} = [\ln(u)]_1^2 = \ln 2$	
Mathematical behaviours	**Marks**
• calculates du/dx correctly • substitutes into integral changing the limits appropriately • integrates the expression correctly	1 1 1

Question 2 (d) **(3 marks)**

Solution	
If we write $\dfrac{x+2}{(x-1)(x-4)} = \dfrac{A}{x-1} + \dfrac{B}{x-4} = \dfrac{A(x-4)+B(x-1)}{(x-1)(x-4)} = \dfrac{(A+B)x-(4A+B)}{(x-1)(x-4)}$ then we conclude that $A+B=1$ and $4A+B=-2$ whence $A=-1$ and $B=2$. Hence $\displaystyle\int \frac{x+2}{(x-1)(x-4)}dx = \int\left(\frac{2}{x-4} - \frac{1}{x-1}\right)dx = 2\ln\|x-4\| - \ln\|x-1\| + C$	
Mathematical behaviours	**Marks**
• writes the correct form of the partial fractions • deduces the correct values of the constants A and B • evaluates the integral correctly • no penalty for omitting the arbitrary constant	1 1 1

Question 3(a) **(4 marks)**

Solution	
Assuming $a = 2$, the system of equations reduces to $x + 0.5y + 0.5z = 0.5 \quad 0.5y - 4.5z = -0.5 \quad 0.5y - 2.5z = 1.5$ (*) and then to $x + 0.5y + 0.5z = 0.5 \quad y - 9z = -1 \quad 2z = 2$ (**) So $z = 1$ Back substitution gives $y = 8$ and $x = -4$	
Mathematical behaviours	**Marks**
• eliminates first variable (*) • eliminates second variable (**) • back substitutes for second variable • back substitutes for first variable	1 1 1 1

Question 3(b) **(2 marks)**

Solution	
The system of equations reduces to $x + 0.5y + 0.5z = 0.5 \quad 0.5y - 4.5z = -0.5 \quad 0.5y + (a - 4.5)z = 1.5$ and then to $x + 0.5y + 0.5z = 0.5 \quad y - 9z = -1 \quad az = 2$ (*) There is no solution if $a = 0$	
Mathematical behaviours	**Marks**
• eliminates first two variables (*) • obtains correct answer	1 1

Question 4(a)(i) **(1 mark)**

Solution	
$(5+2i)^2 = 25+10i+10i+(2i)^2 = 21+20i$	
Mathematical behaviours	Marks
• verifies the result	1

Question 4(a)(ii) **(2 marks)**

Solution	
The equation $z^2 - z - 5(1+i) = 0$ has solutions $$z = \frac{1}{2}\left(1 \pm \sqrt{1+20(1+i)}\right) = \frac{1}{2}\left(1 \pm \sqrt{21+20i}\right) = \frac{1}{2}\left(1 \pm (5+2i)\right)$$ so that $z = 3+i$ or $z = -2-i$.	
Mathematical behaviours	Marks
• uses quadratic formula identifying the appropriate square root • deduces the two solutions	1 1

Question 4(b) **(2 marks)**

Solution	
Since $$\frac{4(1+i)}{(1-i)} = \frac{4(1+i)(1+i)}{(1-i)(1+i)} = \frac{4(2i)}{2} = 4i = 4\exp(i\pi/2)$$ we conclude that $w = 2\exp(i\pi/4)$	
Mathematical behaviours	Marks
• multiplies by the complex conjugate • writes w in polar form	1 1

Question 4(c) **(2 marks)**

Solution	
As $w = 2\exp(i\pi/4)$ this means that if the complex number z is multiplied by w the effect is to double the distance of the point in the Argand diagram from the origin. Moreover, the line joining the origin to z is rotated anticlockwise through an angle $\pi/4$	
Mathematical behaviours	Marks
• recognises the effect is to double the distance from O • comments that there is an anticlockwise rotation through $\pi/4$	1 1

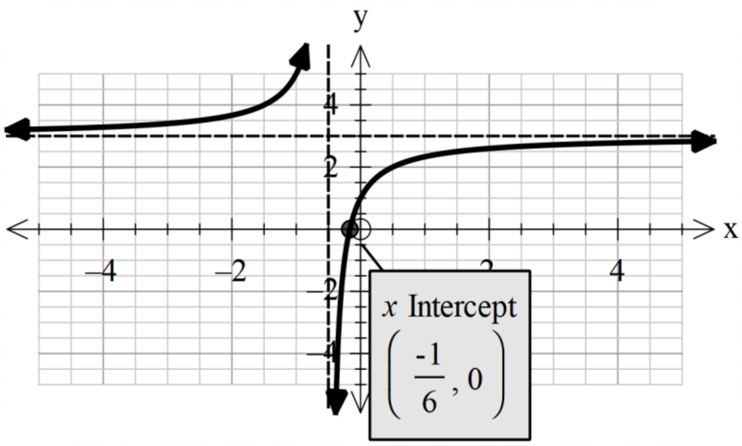
Question 5(a) **(2 marks)**

Solution	
The general equation of a plane is $\boldsymbol{r}.\boldsymbol{n} = c$, so $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{2k}$ is a normal to the plane. (*) Since $\|2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$, $2/3\boldsymbol{i} + 1/3\boldsymbol{j} - \boldsymbol{2}/3\boldsymbol{k}$ is a unit normal to the plane. (So too is $-2/3\boldsymbol{i} + 1/3\boldsymbol{j} - \boldsymbol{2}/3\boldsymbol{k}$.)	
Mathematical behaviours	**Marks**
• obtains a normal to the plane (*)	1
• divides by the length to get a unit vector	1

Question 5(b) **(3 marks)**

Solution	
If $A(a, b, c)$ is the point on the plane closest to the origin then $2a + b - 2c = 18$ and $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is normal to the plane. So $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} = t(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$ for some real number t. So $2(2t) + t - 2(-2)t = 18$, i.e. $9t = 18$ and so $t = 2$. So the coordinates of A are $(4,2,-4)$	
Mathematical behaviours	**Marks**
• recognizes that OA is normal to the plane	1
• solves for the parameter t	1
• obtains correct answer	1

Question 6(a) **(4 marks)**

Solution


Mathematical behaviours	Marks
• indicates asymptote at $x=-\frac{1}{2}$ correctly	1
• indicates asymptote at $y=3$ correctly	1
• indicates y-intercept at (0,1) and x-intercept at $\left(-\frac{1}{6},0\right)$	1
• correct shape of the curve	1

Question 6(b) **(2 marks)**

Solution

$$6x+1=(kx+1)(2x+1)$$

$$\Rightarrow \frac{(6x+1)}{(2x+1)}=(kx+1)$$

$\Rightarrow\ y=\frac{6x+1}{2x+1}$ and $y=kx+1$ will intersect at 2 points with positive x-coordinates

i.e.

$$6x+1=(kx+1)(2x+1)$$

has 2 non-negative roots if $0<k<4$

Mathematical behaviours	Marks
• states that $k>0$	1
• states that $k<4$	1

Question 7(a) **(2 marks)**

Solution	
$\tan\theta = \frac{5}{x}$ $\Rightarrow x = \frac{5}{\tan\theta}$ or $x = \frac{5\cos\theta}{\sin\theta}$	
Mathematical behaviours	Marks
• uses tangent ratio to form a correct equation • expresses x in terms of θ correctly	1 1

Question 7(b) **(2 marks)**

Solution	
$\frac{dx}{d\theta} = \frac{\sin\theta(-5\sin\theta) - 5\cos\theta(\cos\theta)}{\sin^2\theta} = \frac{-5(\sin^2\theta + \cos^2\theta)}{\sin^2\theta} = -\frac{5}{\sin^2\theta}$	
Mathematical behaviours	Marks
• uses quotient rule to differentiate $\frac{5\cos\theta}{\sin\theta}$ correctly • states correct expression in simplest form	1 1

Question 7(c) **(4 marks)**

Solution	
When $x = 5$, $\frac{dx}{dt} = -3$ (towards O) and $\tan\theta = \frac{5}{5} = 1$ so that $\theta = \pi/4$ Then $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{\sin^2\left(\frac{\pi}{4}\right)}{-5} \times (-3) = \frac{3}{5} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{10}$ radians/sec	
Mathematical behaviours	Marks
• identifies $\frac{dx}{dt} = -3$ • determines value of θ correctly when $x = 5$ • identifies correct expression for $\frac{d\theta}{dx}$ • uses chain rule correctly with appropriate substitution to evaluate the value of $\frac{d\theta}{dt}$	1 1 1 1

Question 8(a) **(2 marks)**

Solution	
This is false. A confidence interval may contain none of the underlying population. (For example, if the population consists of an equal number of $0's$ and $1's$, and if the sample is large enough, then a 90% confidence interval will be a small interval of numbers near 0.5 and will contain neither 0 nor 1.	
Mathematical behaviours	Marks
• gives correct answer • gives a valid reason	1 1

Question 8(b) **(2 marks)**

Solution	
This is false. The probability that any one of the confidence intervals will contain μ is 0.95. But it is possible that NONE of the twenty confidence intervals will contain μ.	
Mathematical behaviours	Marks
• gives correct answer • gives a valid reason	1 1

Question 8(c) **(2 marks)**

Solution	
This is true. The width of the confidence interval is proportional to $1/\sqrt{n}$. So to halve the width the sample size must be reduced by a factor of 4	
Mathematical behaviours	Marks
• gives correct answer • gives a valid reason	1 1